Table 2 Wall values of the temperatuie function g

Pr	$g_0(0)$	$g_1(0)$	$g_2(0)$
0 6	0 386432	−0 0199978	0 0108439
0 7	0 417858	−0 0146513	0 00763153
0 8	0 447011	−0 00955783	0 00480933
0 9	0 474293	−0 00468295	0 00228608
1 0	0 500000	0 000000	0 000000
1 1	0.524358	0.00451234	−0.00209281

spondingly, the recovery factor for the cylinder is identical to that for the flat plate When the Prandtl number departs from unity, the sign of the transverse curvature effect is different depending upon whether $Pr > 1$ or $Pr < 1$ For moderate values of ξ (<1), the recovery factor for the cylinder falls below that for the flat plate when $Pr < 1$, whereas the opposite trend is in evidence when $Pr > 1$ The magnitude of the transverse curvature effect increases as the Prandtl number deviates more and more from unity

It is interesting to compare the foregoing results with corresponding information in the literature From the work of Seban and Bond, it appears that the only available result is the value $g_1(0) = 0\ 0034$ for $Pr = 0\ 715$ Upon comparing with the listing of Table 1, it is seen that this $g_1(0)$ is both of opposite sign and smaller by a factor of four relative to the present results Such a disparity may be due, at least in part, to inaccuracies in the velocity profiles that were used as input in the Seban-Bond solutions

The distribution of g_0 as a function of η is identical to the flat-plate solution that may be found in standard references such as Schlichting [4] Space limitations preclude a complete presentation of the g_1 and g_2 profiles; however, representative results are shown in Fig 1 for $Pr = 0\ 6$ and $1\ 1$ As is usual for perturbation functions, the curve for g_2 crosses the zero line one more time than does the curve for g_1 Except in the neighborhood of the surface (i e , $\eta = 0$), the profiles for $Pr = 0\ 6$ and $1\ 1$ are of the same general form However, there are distinct differences near $\eta = 0$, and these are reflected in the adiabatic-wall temperature results

It may also be mentioned that an attempt was made to derive recovery factors at large downstream distances from the leading edge of the cylinder by applying the method of Bourne and Davies,[5] but without success

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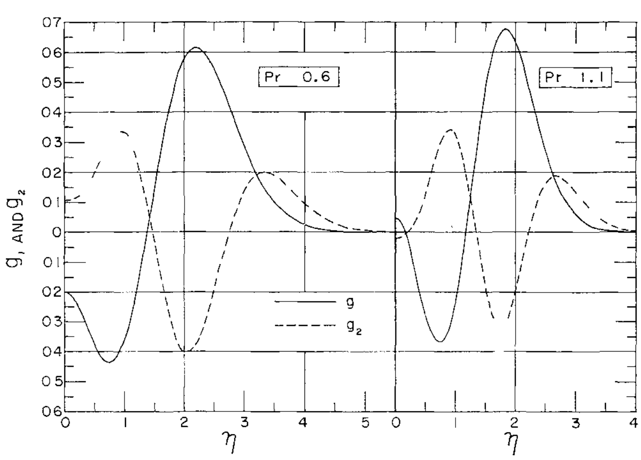

Fig 1 Repiesentative profiles of the temperatuie functions g_1 and g_2

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Numerical Stability of the Three-Dimensional Method of Characteristics

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Introduction

THE numerical calculation of hyperbolic flow fields in three independent variables has been considered recently by several authors [1–4] Some fundamental differences occur in generalizing the method of characteristics from two independent variables to three For example, with two independent variables, writing the governing equations in characteristic form reduces the partial differential equations to ordinary differential equations In three independent variables the equations, even in characteristic form, are still partial differential equations

This note considers effects that arise from a second basic difference between problems in two and three independent variables In the two-dimensional case there are two characteristic lines passing through a point (neglecting the streamline), whereas in the three-dimensional case an infinite number of characteristic surfaces pass through a point (a single-parameter family) This introduces a "degree of freedom" in the choice of the net to be used in the three-dimensional problem, which is not present in the two-dimensional case This freedom of choice of the net has led to the investigation of various net configurations Fowell[1] discusses five different net configurations and the various advantages and disadvantages of each It is the purpose of this note to discuss the stability of certain of the net configurations being used at the present time

The Stability Criterion

Consider a set of first-order hyperbolic partial differential equations of the form

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{m} A^k \frac{\partial u}{\partial x^k} \tag{1}$$

where u is a vector of n unknowns, and A^k are real constant coefficient matrices Courant, Friedrichs, and Lewy[5] determined that a necessary condition for convergence of a difference scheme for this equation is that the domain of dependence of the difference scheme must contain the domain of dependence of the differential equation Hahn,[6] using the work of Lax,[7] showed that for simplicial difference schemes (i e , schemes that use a minimum number of points in the initial surface to determine a new point), the Courant-Friedrichs-Lewy condition is sufficient as well as necessary for convergence and, therefore, also for stability

Received November 18, 1963 This work was sponsored by the Air Force Office of Scientific Research under Grant AF-AFOSR-156-63 The numerical computations were carried out at the Massachusetts Institute of Technology Computation Center on an IBM 7094 computer The authors wish to gratefully acknowledge the invaluable assistance of Gilbert Strang of the Massachusetts Institute of Technology Mathematics Department who indicated the probable cause of instability and pointed out the references to the mathematical literature

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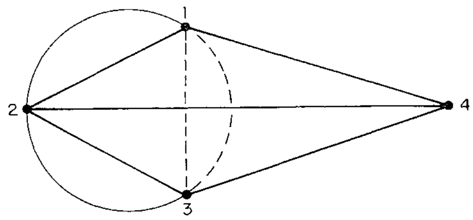

Fig 1 Network A, tetrahedral characteristic line network

The linearized equations for inviscid two-dimensional unsteady flow or steady three-dimensional supersonic flow can be put in the form of Eq (1) Thus, the stability and convergence criterion mentioned in the foregoing paragraph applies at least locally (because of the linearization) to these two types of flow

It is possible to remove the restriction that the criterion applies only locally because of the linearization if the equations and the true solution possess a certain degree of smoothness Strang[8] has shown that if the equations and their solution possess enough continuous derivatives then the convergence of the full nonlinear equations depends on the stability of the linearized equations The existence of the continuous derivatives of the solution cannot be proved before the solution is known, but it might be expected that for "reasonably well-behaved" physical problems the solution will possess the required degree of smoothness

Net Configurations

The stability criterion stated in the foregoing section can be applied to some of the basic finite difference networks proposed for the three-dimensional method of characteristics Fowell discusses and compares the ease of computation for five basic net configurations The first, which he refers to as the tetrahedral characteristic line network (Fig 1), is judged by Fowell to be the most advantageous Unfortunately, when the stability criterion is applied to this network it is found to be unstable Briefly, the net is formed by taking the points, numbered 1–3, in the initial surface and finding the common intersection of the envelopes of their characteristic surfaces (Mach cones) This point of intersection is numbered 4 in Fig 1 A good approximation to the domain of dependence of point 4 in the initial surface is the circle containing points 1–3 This is the domain of dependence for the differential equations The domain of dependence of the difference net is the triangle with vertices at 1–3 (formally termed the convex hull of the difference scheme) The domain of dependence of the difference scheme does not contain the domain of dependence of the differential equations and therefore does not fulfill the stability criterion

Physically, the stability criterion is very logical, because it is not hard to imagine that, in certain areas of a flow field, points lying within the circle but outside the triangle circumscribed by the circle could have a large influence on the new point being calculated However, in this net, effects of points outside the triangle are not taken into account in the numerical calculation so that the solution could be erroneous

To insure stability, the authors propose a modification (shown in Fig 2) to the tetrahedral characteristic line network Starting with the original initial points 1–3, points 5–7 are determined by inscribing a circle in the triangle with vertices at 1–3 The points of tangency between the circle and sides of the triangle are chosen as intermediate initial points, and the flow properties at these points are determined by interpolation between the points 1–3 Points 5–7 are then used in exactly the same way as 1–3 were used in the original network shown in Fig 1 This modified network fulfills the stability criterion, in that the circular domain of dependence of the differential equations falls within the triangular domain of dependence of the difference scheme If the same initial points are used to calculate a new point with both network A and network B, the step size from the initial surface to the new point is smaller for network B Thus, the use of network B requires a larger number of calculations to do a given problem, but it is well known that, in straightforward finite difference techniques, decreasing the step size has a stabilizing influence Numerical results of calculations that compare the two networks are given in the next section

The second net configuration discussed by Fowell is the tetrahedral characteristic surface network This net is also simplicial, but is found to be stable when the stability criterion is applied Tsung[4] used this network and indeed found no instability in his calculations

Three other nets are discussed by Fowell, but none of these are simplicial The physical argument given in the second paragraph of this section would still seem to apply to these networks, however Hence, application of the criterion would seem to give some insight into the stability of the net even though it does not formally apply

It appears that these three networks might be unstable if only the local net configuration is considered However, Moretti[3] has used the network of intersections of reference planes with characteristic surfaces and Butler[2] the pentahedral bicharacteristic line network, and they encountered no instability in their calculations Closer examination of the networks reveals that in each of the calculations the points in the initial surface must be moved around in the initial surface as the solution at the new point is obtained by iteration This requires that the properties at the initial points must be obtained by interpolation in the initial surface The interpolation schemes used by Moretti and Butler are such that they effectively increase the domain of dependence of the difference scheme to the extent that it contains the domain of dependence of the differential equations Hence, the over-all schemes should be stable

Numerical Results

The authors are interested in applying the method of characteristics to two- and three dimensional unsteady flow problems and have made calculations using networks A and B shown in Figs 1 and 2 Network B is exactly the same as A except that one initial interpolation is carried out This interpolation is carried out only once and is not repeated during the iteration process, so that network B still retains the advantages cited by Fowell for network A

As one application of the method of characteristics in two-dimensional unsteady flow, the authors have considered the flow over a blunt body submerged in a supersonic stream Starting from some known initial flow or from some initial approximation, the body surface can be held constant or made to perform some motion and the flow field determined in the mixed subsonic supersonic region for the blunt body It is hoped that interesting steady flows might be obtained by starting from some known flow about a simple blunt body and then warping the body surface to a different shape and allowing the flow field to settle to a new steady flow

A calculation has been made of the flow of a perfect gas about a circular cylinder with its axis normal to a Mach 5 freestream Data on the initial surface were obtained from the work of Belotserkovskii,[9] which was calculated using two strips in the method of integral relations The method

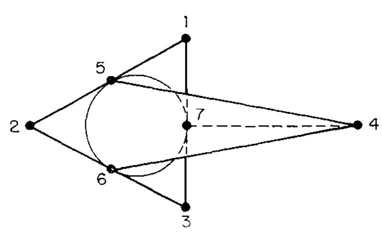

Fig 2 Network B, proposed modified network

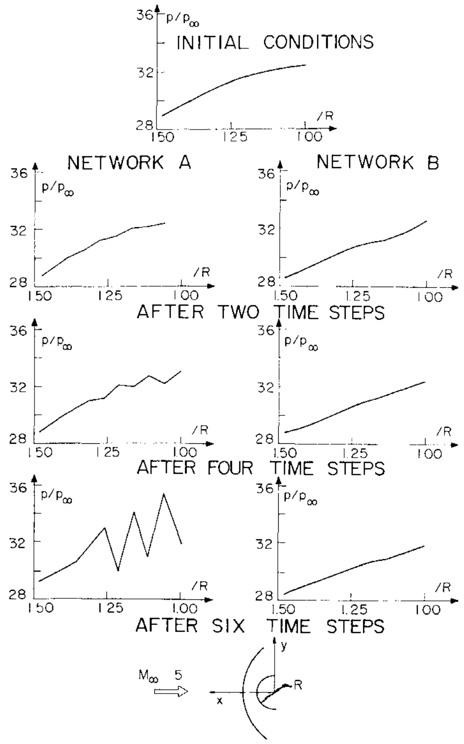

Fig 3 The pressure near the axis of symmetry for a blunt-body flow

of characteristics using network A was first used to calculate the flow with the body surface held constant in time to check that the initial solution would hold steady as the flow was calculated at later times It was found that the pressure along the axis of symmetry was the most sensitive indicator of the onset of instability Figure 3 shows plots of the pressure ratio vs the coordinate parallel to the freestream velocity for points closest to the axis after various time steps in the calculation Note that these points are not located exactly on the axis and do not have the same space coordinates at each step so that the magnitudes of the pressure ratio should not be exactly the same on each plot; however, they do indicate very graphically the onset of instability Network B then was used to repeat the calculation with the results shown on the right side of Fig 3 The instability with network A had grown so large by the eighth step that calculation could not be continued, whereas no instability has been detected while using the modified network B

Conclusions

Previously, it has been tacitly assumed in three-dimensional characteristic calculations that integration on the characteristic surface, or at least along bicharacteristic lines, assures the convergence and stability of the process This is not the case Care must be taken in choosing a net configuration that is stable and convergent The Courant Friedrichs-Lewy condition can be employed as a test for stability and convergence of a finite difference network that is simplicial, and it can be argued physically that this provides a necessary condition for the stability of networks that are not simplicial

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Elastic Stability of Conical Shells under Axisymmetric Pressure Band Loadings

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IN a recent publication,[1] the buckling characteristics of circular cones and cylinders subjected to axisymmetric loadings were investigated Typical results were illustrated for various loadings and compared with previous analytical and experimental studies

This paper is concerned with applying the theoretical equations developed in Ref 1 to assess the stability response of simply supported cones and cylinders under circumferential band loadings One important purpose of the present investigation is to illustrate a simplified method for estimating the critical pressures for circumferential band loadings Another objective is to demonstrate that the predictions based on such a method are satisfactorily correlated with those rigorously obtained by Almroth and Brush[2] for the limiting case of a circumferential band loading that is uniformly distributed and equally spaced between cylinder supports

The value of such a correlation is that it lends credence to the present method of approach and, further, indicates the usefulness of the Donnell-type theory used in Ref 1 for estimating the stability response for arbitrary, axisymmetric loading conditions The analysis now is extended for the simply supported cylinder under a uniformly distributed and equally spaced, circumferential band loading

The equation governing the elastic buckling of circular cylinders subjected to axisymmetric loadings is given by [e g , Ref 1, Eq (28)]

$$\bar{P} = [\bar{\phi}^2 + 1/\bar{\phi}^2]/(\bar{\phi} + \bar{S}/2) \tag{1}$$

where

$$\begin{aligned} \bar{P} &= \bar{p}_c K_2 12(1 - \nu^2)/\lambda^2 \bar{H}^{1/4} E h^3 \\ \bar{\phi} &= (1 + n^2/\lambda^2 a^2)/\bar{H}^{1/4} \\ \bar{S}/2 &= [(K_1/a^2 K_2) - 1]/\bar{H}^{1/4} \\ \bar{H} &= 12\,(1 - \nu^2)/\lambda^4 a^2 h^2 \\ \lambda &= m\pi/L \end{aligned} \tag{2}$$

and

$$\begin{aligned} K_1 &= -\frac{2a^2}{L}\int_0^L \left(\frac{\bar{N}_\xi}{p}\right)\cos^2\lambda\xi d\xi \\ K_2 &= -\frac{2}{L}\int_0^L \left(\frac{\bar{N}_\theta}{p}\right)\sin^2\lambda\xi d\xi \end{aligned} \tag{3}$$

Received November 27, 1963
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